# Uniform Distribution

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These notes are from a series of lectures on uniform distribution that were given by Prof. Vitaly Bergelson at the 2024 Ross Mathematics Program.

## #1 Introduction

**Definition 1.1.** A set $S \subset [0,1]$ is **dense** in $[0,1]$ if for any $0 \leq a < b \leq 1$, $S \cap [a,b] \neq \varnothing$.

Some examples of sets that are dense in $[0,1]$ include

- $[0,1]$ itself, or the open interval $(0,1)$, or $[0,1]$ minus any finite set;
- $\mathbb{Q} \cap [0,1]$ and $[0,1] \setminus \mathbb{Q}$, or the rationals and irrationals in $[0,1]$, respectively;
- the complement of any countable set in $[0,1]$; and
- the **dyadic rationals** (expressible as fractions whose numerators are integers and whose denominators are powers of 2) in $[0,1]$.

**Definition 1.2.** A set $E \subset \mathbb{R}$ has **measure zero** if for any $\varepsilon > 0$, it can be covered by a family of open intervals with total length less than (or equal to) $\varepsilon$.

We will not use measure theory in this course. However, the concept of measure zero is a powerful definition. Recall a theorem of Lebesgue below.

**Theorem 1.3** (Lebesgue)

A bounded function $f : [a,b] \to \mathbb{R}$ is **Riemann integrable** iff its set of points of discontinuity has measure zero.

**Proposition 1.4**

Any finite or countable set has measure zero.

**Proof.** Let $\varepsilon > 0$. If $S = \{x_1, \ldots, x_n\}$ is finite, we take $I_n = (x_n - \varepsilon/2n, x_n + \varepsilon/2n)$ to be the open interval covering each $x_n$. Then $\sum |I_n| = \varepsilon$.

If $S = \{x_1, x_2, \ldots\}$ is countable, we take $I_n = (x_n - \varepsilon/2^{n+1}, x_n + \varepsilon/2^{n+1})$ to be the open interval covering each $x_n$. Then $\sum |I_n| = \varepsilon$ once again. □

Now, a natural question is whether or not uncountable sets can have measure zero.

**Exercise 1.5.** Show that $[0, 1]$ is not of measure zero.

## #2 The Cantor set

We continue exploring these notions by introducing the ubiquitous Cantor set — in particular, the "classical" or "middle thirds" Cantor set, denoted $\mathcal{C}$.

We construct $\mathcal{C}$ by repeatedly removing the open middle thirds of intervals in $[0, 1]$ :

- Define $K_1$ to be $[0, 1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right)$ .
- Define $K_2$ to be $K_1 \setminus \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right)$ .
- Continue this process infinitely, obtaining $K_{n+1}$ by removing the middle thirds of all remaining intervals in $K_n$.

**Definition 2.1.** Formally, we define the **Cantor set** to be $\mathcal{C} = \bigcap_{n=1}^{\infty} K_n$.

Note that $\mathcal{C}$ is nonempty (for instance, it contains 0) and infinite (in fact, it contains all the endpoints of the removed open intervals).

**Definition 2.2.** A set $S \subset \mathbb{R}$ is **open** if for any $x \in S$, there exists an open interval $(a, b)$ containing $x$ such that $(a, b) \subset S$. A set $S \subset \mathbb{R}$ is **closed** if its complement is open.

> **Remark 2.3.** "Not open" is not the same as "closed"! Many sets are neither open nor closed (and some can be both open *and* closed).

**Exercise 2.4.** Prove that any open set in $\mathbb{R}$ is a disjoint union of finitely or countably many open intervals. (**Question:** do we have a similar characterization in $\mathbb{R}^n$?)

We claim some properties of $\mathcal{C}$:

- $\mathcal{C}$ is (uncountably) infinite.
- $\mathcal{C}$ is closed.
- $\mathcal{C}$ has measure zero, which we denote as $m(\mathcal{C}) = 0$.

**Exercise 2.5.** Show that $m(\mathcal{C}) = 0$. (**Sketch:** we know that $\mathcal{C} \subset K_n$ for all $n$, and we can choose $K_n$ with arbitrarily small total length.)

**Exercise 2.6.** Show that $\sum_{j=1}^{\infty} \ell(I_j) = 1$, where $I_j$ are the intervals removed from $[0, 1]$ to form $\mathcal{C}$. (**Question:** if the removed intervals have total length 1, then *what's left?*)

We also introduce the concept of "generalized" Cantor sets, where we continue to remove middle intervals in $[0, 1]$ without strict control over the length of these removals.

**Definition 2.7.** Let a **generalized Cantor set** be $B = \bigcap_{n=1}^{\infty} A_n$, where $A_n$ are the sets with middle intervals removed. We shall abbreviate this as **GCS**.

We consider the series $\sum a_n$, where $A_1$ removes 1 interval of length $a_1$, then $A_2$ removes 2 intervals of length $a_2/2$ each, then $A_3$ removes 4 intervals of length $a_3/4$ each, and so on.

**Exercise 2.8.** Show that when $\sum a_n < 1$, $B$ does not have measure zero.

We may wonder about the properties of GCSs — but notice that we have not yet fully understood the properties of the classical Cantor set $\mathcal{C}$. An alternative definition becomes useful (equivalence of these definitions is left as an **Exercise**.)

**Definition 2.9.** We claim that $\mathcal{C} = \left\{t \in [0,1] : t = \sum_{i=1}^{\infty} t_i/3^i,\ t_i \in \{0,2\}\right\}$; that is, $\mathcal{C}$ is the set of all $t \in [0,1]$ whose **ternary expansions** do not contain 1.

> **Remark 2.10.** How do we know that ternary expansion is a good $(1-1)$ encoding? Consider analagous "issues" in decimal expansion, where two encodings may refer to the same number.

This definition elucidates the uncountability of $\mathcal{C}$: we can map each element of $\mathcal{C}$ to an infinite sequence on two symbols (0 and 2), and there are uncountably many such sequences (consider Cantor's diagonalization argument).

For GCSs, we no longer have strict control over the length of removals, so we cannot use ternary expansion; we can, however, encode elements of $B$ with **itineraries**. Informally, if $x \in B$ is on the "left side" of a middle interval removed in $A_n$, then the $n$th entry of the itinerary encoding $x$ is $L$. We similarly use $R$ for the "right side" of removed intervals.

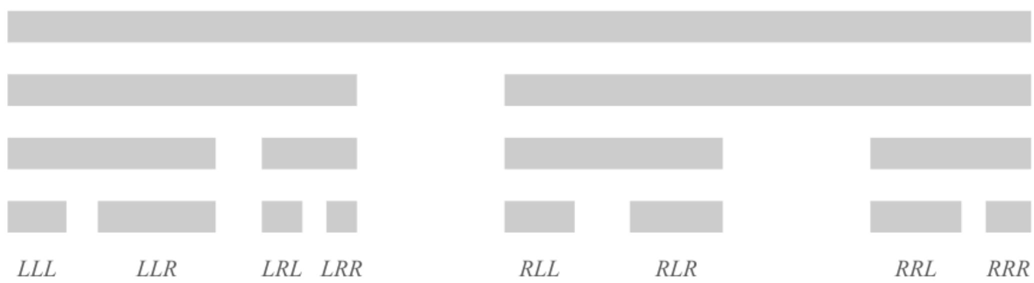


**Figure 2.11.** *The first three entries of itineraries encoding elements of $B$ (a GCS).*

The similarity of the encodings used for $\mathcal{C}$ and $B$ raises the question of whether the two sets are isomorphic. Topologically, we consider **homeomorphisms**.

**Definition 2.12.** Two metric spaces $X$ and $Y$ are **homeomorphic** if there exists a two-way continuous bijection $f : X \to Y$. We denote this by $X \sim Y$.

**Exercise 2.13.** Prove that $\mathcal{C}$ is homeomorphic to any GCS $B$; that is, $\mathcal{C} \sim B$.

> **Remark 2.14.** We must first consider the topology of $\mathcal{C}$ and $B$ in order for us to discuss homeomorphisms; we will take the Euclidean metric from $\mathbb{R}$.

We introduce two more properties of the Cantor set:

- $\mathcal{C}$ is nowhere dense.
- $\mathcal{C}$ is "arithmetically large".

**Definition 2.15.** A set $S \subset [0,1]$ is **nowhere dense** or **rare** in $[0,1]$ if it is not dense in any nonempty open subset of $[0,1]$.

**Exercise 2.16.** Show that $\mathcal{C} + \mathcal{C} = \{x + y \mid x, y \in \mathcal{C}\} = [0,2]$ and $\mathcal{C} - \mathcal{C} = [-1,1]$.

**Remark 2.17.** We see that $\mathcal{C}$ is "small" in some senses and "large" in other senses.

# #3 A number theoretic digression

We spend some time discussing interesting results in combinatorial number theory.

**Definition 3.1.** The **upper density** of a set $A \subset \mathbb{N}$ is defined as

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1,\dots,N\}|}{N}.$$

**Theorem 3.2** (Furstenberg–Sárközy)
If $A \subset \mathbb{N}$ has positive upper density (that is, $\overline{d}(A) > 0$), then there exist $x, y \in A$ and $n \in \mathbb{N}$ such that $x - y = n^2$.

Equivalently, for any set $A \subset \mathbb{N}$ with positive upper density, its difference set $A - A$ contains the square of a positive integer $n$. In fact, we claim that $A - A$ contains $n^k$ for any integer $k > 2$ (though perhaps a different $n$).

What else can we replace $n^k$ with? For instance, consider polynomials $p$ with no constant terms, or $p(0) = 0$. We claim that $n^2 - 1$ works, but $n^2 + 1$, $n!$, $2^n$, and primes do not.

**Exercise 3.3.** Show that if $A \subset \mathbb{N}$ has positive upper density, then its difference set $A - A$ has bounded gaps. Such sets are called **syndetic**.

**Exercise 3.4.** Show that there exists $A \subset \mathbb{N}$ such that $\overline{d}(A)$ and $\overline{d}(\mathbb{N} - A)$ are both 1.

**Exercise 3.5.** Prove that if $A \subset \mathbb{N}$ is syndetic, then it has upper density greater than 1.

**Exercise 3.6.** Give an example of a set $A \subset \mathbb{N}$ with positive upper density such that its sum set $A + A$ is *not* syndetic.

Recalling **Furstenberg–Sárközy**, we may wonder how many $n$ satisfy $n^2 = x - y \in A - A$. It turns out that there are infinitely many such $n$, and the set $S$ of all such $n$ has positive upper density. In fact, $S$ is syndetic!

# #4 Uniform distribution

**Definition 4.1** (UD1). A sequence $(x_n) \subset [0,1]$ is called **uniformly distributed** or **uniformly dense** if for any $0 \le a < b \le 1$, we have

$$\lim_{N\to\infty} \frac{\#\{1 \le n \le N : x_n \in (a,b)\}}{N} = b - a. \qquad \text{(Weyl, 1916)}$$

It is clear that a uniformly dense sequence is dense. If not, then there would exist some subinterval $(a,b)$ containing no elements of $(x_n)$, but $b - a \neq 0$. It is not clear, however, how this miraculous property arises: the sequence must spend the "correct amount of its life" in each of uncountably many subintervals. (**Question:** is this even possible?)

**Exercise 4.2.** Order $\mathbb{Q} \cap [0,1]$ to obtain a uniformly distributed sequence.

**Definition 4.3.** By $C_{\mathbb{R}}[0,1]$ or simply $C[0,1]$, we denote the set of continuous functions $f : [0,1] \to \mathbb{R}$. Similarly, $C_{\mathbb{C}}[0,1]$ is the set of continuous functions $f : [0,1] \to \mathbb{C}$.

To obtain Bergelson's favorite metric space, we equip $C[0,1]$ with the metric

$$d(f,g) := \max_{x\in[0,1]} |f(x) - g(x)|.$$

Recall that a **metric** on a space $X$ should satisfy some axioms (for $a, b, c \in X$):

- Zero: $d(a,a) = 0$.
- Positivity: if $a \neq b$, then $d(a,b) > 0$.
- Symmetry: $d(a,b) = d(b,a)$.
- Triangle inequality: $d(a,b) \le d(a,c) + d(b,c)$.

**Exercise 4.4.** Verify that $d(f,g)$ is indeed a metric on $C[0,1]$.

We also introduce two new definitions of uniform distribution.

**Definition 4.5** (UD2). A sequence $(x_n)$ is **uniformly distributed** if for any $f \in C[0,1]$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx.$$

**Definition 4.6.** We denote by $R[0,1]$ the set of Riemann integrable functions on $[0,1]$.

**Definition 4.7** (UD3). A sequence $(x_n)$ is **uniformly distributed** if for any $f \in R[0,1]$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)\,dx.$$

Note that **(UD1)** and **(UD2)** are special cases of **(UD3)**. We call **(UD2)** the Ergodic Principle, where "time average" = "space average".

**Exercise 4.8.** Show that all three definitions above are equivalent.

We make the useful observation that in order to verify **(UD2)** or **(UD3)**, it suffices to check these formulas for a dense subset of functions in $C[0,1]$ or $R[0,1]$.

**Exercise 4.9.** Verify the observation above.

Informally, just a dense subset of $[0,1]$ should be able to "approximate" any real number in the interval, a dense subset of functions in $C[0,1]$ should be able to "approximate" any function in the space. We formalize this notion with another definition.

**Definition 4.10.** A set $S \subset C[0,1]$ is **dense** in $C[0,1]$ if for any $\varepsilon > 0$ and any $f \in C[0,1]$, there exists some $g \in S$ such that $d(g,f) < \varepsilon$.

**Exercise 4.11** (Reformulation of **Exercise 4.2**)**.** Any countable dense subset $S \subset [0,1]$ can be ordered into a uniformly distributed sequence.

**Definition 4.12.** A metric space $X$ is **separable** if it has a dense countable subset.

We cite some additional theorems and examples in order to help us identify a suitable countable dense subset of $C[0,1]$.

**Theorem 4.13** (Stone–Weierstrass)

Every function in $C[0,1]$ can be uniformly approximated (arbitrarily closely) by a polynomial function. That is, for any $\varepsilon > 0$ and any $f \in C[0,1]$, there exists a polynomial $g$ such that $d(f,g) < \varepsilon$.

**Corollary 4.14**

The space $C[0,1]$ is separable.

**Exercise 4.15.** Show that the **rational polynomials**, or polynomials with rational coefficients, are countable and dense in $C[0,1]$. (Might this help us prove **Corollary 4.14**?)

**Proposition 4.16**

For any $\alpha \notin \mathbb{Q}$, the sequence $\big(n\alpha \pmod 1\big)$ or $\big(n\alpha - \lfloor n\alpha \rfloor\big)$ is dense in $[0,1]$.

**Exercise 4.17.** Let us denote $n\alpha \pmod 1$ by $[n\alpha]$. Is the sequence defined by ordered pairs $x_n = \big([n\sqrt{2}], [n\sqrt{3}]\big)$ dense in $[0,1] \times [0,1]$?

**Exercise 4.18.** What about for general $x_n = \big([n\alpha], [n\beta]\big)$ with $\alpha, \beta \notin \mathbb{Q}$?

**Exercise 4.19.** Correct the proof of **Proposition 4.16** presented in lecture.